OPERATIONS RESEARCH

Chapter 1
Linear Programming Problem

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1.0 Introduction

Linear programming (LP) is a popular tool for solving optimization problems of special kind. In 1947, George Bernard Dantzig developed an efficient method, the simplex algorithm, for solving linear programming problem (LPP). Since the development of the simplex algorithm, LP has been used to solve optimization problems in industries as diverse as banking, education, forestry, petroleum, manufacturing, and trucking. The most common problem in these industries involves allocation of limited resources among competing activities in the best possible (optimal) way. Real world situations where LP can be applied are thus diverse, ranging from the allocation of production facilities to products to the allocation of national resources to domestic needs, from portfolio selection to the selection of shipping patterns, and so on. In this unit, we will discuss the mathematical formulation of LPP, the graphical method for solving two-variable LPP, and simplex algorithm, duality, dual simplex and revised simplex methods for solving LPP of any number of variables.

MODULE - 1: Mathematical Formulation of LPP and Graphical Method for Solving LPP

1.1 Mathematical Formulation of LPP

There are four basic components of an LPP:

• *Decision variables* - The quantities that need to be determined in order to solve the LPP are called decision variables.

• *Objective function* - The linear function of the decision variables, which is to be maximized or minimized, is called the objective function.

• *Constraints* - A constraint is something that plays the part of a physical, social or financial restriction such as labor, machine, raw material, space, money, etc. These limits are the degrees to which an objective can be achieved.

• *Sign restriction* - If a decision variable x_i can only assume nonnegative values, then we use the sign restriction $x_i \geq 0$. If a variable x_i can assume positive, negative or zero values, then we say that x_i is unrestricted in sign.

A **linear programming problem (LPP)** is an optimization problem in which
(i) the linear objective function is to be maximized (or minimized);
(ii) the values of the decision variables must satisfy a set of constraints where each constraint must be a linear equation or linear inequality;
(iii) A sign restriction must be associated with each decision variable.

Two of the most basic concepts associated with LP are **feasible region** and **optimal**

solution.

• *Feasible region* - The **feasible region** for an LPP is the set of all points that satisfy all the constraints and sign restrictions.

• *Optimal solution* - For a maximization problem, an optimal solution is a point in the feasible region with the largest value of the objective function. Similarly, for a minimization problem, an **optimal solution** is a point in the feasible region with the smallest value of the objective function.

1.1.1 General Linear Programming Problem

A general linear programming problem can be mathematically represented as follows:

$$
\begin{aligned}
&\text{Maximize (or Minimize) } Z = c_1x_1 + c_2x_2 + ... + c_nx_n \\
&\text{subject to,} \\
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1j}x_j + ... + a_{1n}x_n \ \ (\leq, =, \geq) \ \ b_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2j}x_j + ... + a_{2n}x_n \ \ (\leq, =, \geq) \ \ b_2 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + ... + a_{ij}x_j + ... + a_{in}x_n \ \ (\leq, =, \geq) \ \ b_i \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mj}x_j + ... + a_{mn}x_n \ \ (\leq, =, \geq) \ \ b_m \\
&\text{and } x_1, x_2, ..., x_n \geq 0
\end{aligned}
$$

The above can be written in compact form as

$$\text{Maximize (or Minimize) } Z = \sum_{j=1}^{n} c_j x_j \tag{1.1}$$

subject to,

$$\sum_{j=1}^{n} a_{ij}x_j \ \ (\leq, =, \geq) \ \ b_i; \ \ i = 1, 2, ..., m \tag{1.2}$$

$$x_j \geq 0; \ \ j = 1, 2, ..., n. \tag{1.3}$$

The problem is to find the values of x_j's that optimize (maximize or minimize) the objective function (1.1). The values of x_j's must satisfy the constraints (1.2) and non-negativity restrictions (1.3). Here, the coefficients c_j's are referred to as *cost coefficients* and a_{ij}'s as *technological coefficients*; a_{ij} represents the amount of the ith resource consumed per unit variable x_j and b_i, the total availability of the ith resource.

Example 1.1: An oil company owns two refineries – refinery A and refinery B. Refinery A is capable of producing 20 barrels of petrol and 25 barrels of diesel per day. Refinery B is capable of producing 40 barrels of petrol and 20 barrels of diesel per day. The company requires at least 1000 barrels of petrol and at least 800 barrels of diesel. If it costs Rs. 300 per day to operate refinery A and Rs. 500 per day to operate refinery B, how many days should each refinery be operated by the company so as to minimize costs? Formulate this problem as a linear programming model.

Solution: Let x_1 and x_2 be the numbers of days the refineries A and B are to be operated, respectively. Our objective is to minimize $Z = 300x_1 + 500x_2$. The total amount of petrol produced is $20x_1 + 40x_2$. As at least 1000 barrels of petrol is required, we have the inequality $20x_1 + 40x_2 \geq 1000$. Similarly, the total amount of diesel produced is $25x_1 + 20x_2$. As at least 800 barrels of diesel is required, we have the inequality $25x_1 + 20x_2 \geq 800$. Hence our linear programming model is

$$\begin{aligned}
&\text{Minimize } Z = 300x_1 + 500x_2\\
&\text{subject to}\\
&20x_1 + 40x_2 \geq 1000\\
&25x_1 + 20x_2 \geq 800\\
&x_1,\ x_2 \geq 0.
\end{aligned}$$

Example 1.2: In a given factory, there are three machines M_1, M_2 and M_3 used in making two products P_1 and P_2, respectively. One unit of P_1 occupies machine M_1 for 5 minutes, machine M_2 for 3 minutes and machine M_3 for 4 minutes, respectively. The corresponding figures for one unit of P_2 are 1 minute for machine M_1, 4 minutes for machine M_2 and 3 minutes for machine M_3, respectively. The net profit for 1 unit of P_1 is Rs. 30 and for 1 unit of P_2 is Rs. 20 (independent of whether the machines are used to full capacity or not). What production plan gives the most profit? Formulate the problem as a linear programming problem.

Solution: Let x_1 = number of units of P_1 produced per hour and x_2 = number of units of P_2 produced per hour. Then the total profit from these two products is $z = 30x_1 + 20x_2$. Now, x_1 units of P_1 occupies $x_1/12$ hours at machine M_1, $x_1/20$ hours at machine M_2 and $x_1/15$ hours at machine M_3. Similarly, x_2 units of P_2 occupies $x_2/60$ hours at machine M_1, $x_2/15$ hours at machine M_2 and $x_2/20$ hours at machine M_3. Therefore, we must have

$\frac{x_1}{12} + \frac{x_2}{60} \leq 1$ or, $5x_1 + x_2 \leq 60$ for machine M_1

$\frac{x_1}{20} + \frac{x_2}{15} \leq 1$ or, $3x_1 + 4x_2 \leq 60$ for machine M_2

$\frac{x_1}{15} + \frac{x_2}{20} \leq 1$ or, $4x_1 + 3x_2 \leq 60$ for machine M_3

Thus the programming model for the production plan is

$$\begin{aligned} &\text{Maximize } \; z = 30x_1 + 20x_2 \\ &\text{subject to} \\ &5x_1 + x_2 \leq 60 \\ &3x_1 + 4x_2 \leq 60 \\ &4x_1 + 3x_2 \leq 60 \\ &x_1, \; x_2 \geq 0. \end{aligned}$$

1.2 LP Solution

1.2.1 Some Terminologies for Solution

• *Closed half plane* - A linear inequality in two variables is known as a half plane. The corresponding equality or the line is known as the boundary of the half plane. The half plane along with its boundary is called a closed half plane.

• *Convex set* - A set is convex if and only if, for any two points on the set, the line segment joining those two points lies entirely in the set. Mathematically, A set S is said to be convex if for all $\mathbf{x}, \mathbf{y} \in S$, $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S$, for all $\lambda \in [0,1]$.

For example, the set $S = \{(x,y) : 3x + 2y \leq 12\}$ is convex because for two points (x_1, y_1) and $(x_2, y_2) \in S$, it is easy to see that $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in S$ for all $\lambda \in [0,1]$.

On the other hand, the set $S=\{(x,y) : x^2 + y^2 \geq 16\}$ is not convex. Note that the two points (4,0) and (0,4) $\in S$ but $\lambda(4,0) + (1-\lambda)(0,4) \notin S$ for $\lambda = 1/2$.

• *Convex polygon* - A convex polygon is a convex set formed by the intersection of a finite number of closed half planes.

• *Extreme points* - The extreme points of a convex polygon are the points of intersection of the lines bounding the feasible region.

• *Feasible solution (FS)* - Any non-negative solution which satisfies all the constraints is known as a feasible solution of the problem.

• *Basic solution (BS)* - For a set of m simultaneous equations in n variables $(n > m)$ in an LP problem, a solution obtained by setting $(n-m)$ variables equal to zero and solving for remaining m equations with m variables is called a basic solution. These m variables are called *basic variables* and $(n-m)$ variables are called *non-basic variables*.

• *Basic feasible solution (BFS)*- A basic solution to an LP problem is called basic feasible solution (BFS) if it satisfies all the non-negativity restrictions. A BFS is called **degenerate** if the value of at least one basic variable is zero, and **non-degenerate** if the values of all basic variables are non-zero and positive.

• *Optimal basic feasible solution* - A basic feasible solution is called optimal, if it optimizes (maximizes or minimizes) the objective function.

The objective function of an LPP has its optimal value at an extreme point of the convex polygon generated by the set of feasible solutions of the LPP.

• *Unbounded solution* - An LPP is said to have unbounded solution if its solution can be made infinitely large without violating any of the constraints.

1.2.2 Some Important Results

Theorem 1.1: *A hyperplane is a convex set.*

Proof: Consider the hyperplane $S = \{\mathbf{x} : \mathbf{cx} = z\}$. Let $\mathbf{x_1}$ and $\mathbf{x_2}$ be two points in S. Then $\mathbf{cx_1} = z$ and $\mathbf{cx_2} = z$. Now, let a point $\mathbf{x_3}$ be given by the convex combination of $\mathbf{x_1}$ and $\mathbf{x_2}$ as $\mathbf{x_3} = \lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2},\ \ 0 \le \lambda \le 1$. Then

$$\begin{aligned}\mathbf{cx_3} &= \mathbf{c}\{\lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2}\}\\ &= \lambda\mathbf{cx_1} + (1-\lambda)\mathbf{cx_2}\\ &= \lambda z + (1-\lambda)z\\ &= z\end{aligned}$$

Therefore, $\mathbf{x_3}$ satisfies $\mathbf{cx} = z$ and hence $\mathbf{x_3} \in S$. $\mathbf{x_3}$ being the convex combination of $\mathbf{x_1}$ and $\mathbf{x_2}$ in S, S is a convex set. Thus a hyperplane is a convex set.

Theorem 1.2: *Intersection of two convex sets is a convex set.*

Proof: Let S_1 and S_2 be two convex sets and $S = S_1 \cap S_2$. Let $\mathbf{x_1}$ and $\mathbf{x_2}$ be two points in S. Since $\mathbf{x_1}, \mathbf{x_2} \in S_1$ and S_1 is convex, therefore, $\lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2} \in S_1$ for $0 \le \lambda \le 1$. Again, since $\mathbf{x_1}, \mathbf{x_2} \in S_2$ and S_2 is convex, therefore, $\lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2} \in S_2$ for $0 \le \lambda \le 1$. Thus $\lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2} \in S$ for $\mathbf{x_1}, \mathbf{x_2} \in S$ and $\lambda \in [0,1]$. Hence $S = S_1 \cap S_2$ is a convex set.

Theorem 1.3: *The set of all feasible solutions of an LPP is a convex set.*

Proof: Consider an LPP whose constraints are $\mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$. Let S be the set of feasible solutions of the LPP and $\mathbf{x_1}, \mathbf{x_2} \in S$. Then $\mathbf{Ax_1} = \mathbf{b},\ \mathbf{x_1} \geq \mathbf{0}$ and $\mathbf{Ax_2} = \mathbf{b},\ \mathbf{x_2} \geq \mathbf{0}$. Let $\mathbf{x_3} = \lambda \mathbf{x_1} + (1-\lambda)\mathbf{x_2},\ \ 0 \leq \lambda \leq 1$. Then

$$\begin{aligned} \mathbf{Ax_3} &= \mathbf{A}\{\lambda \mathbf{x_1} + (1-\lambda)\mathbf{x_2}\} \\ &= \lambda \mathbf{Ax_1} + (1-\lambda)\mathbf{Ax_2} \\ &= \lambda b + (1-\lambda) b \\ &= b \end{aligned}$$

Also, $\mathbf{x_3} \geq \mathbf{0}$ as $\mathbf{x_1} \geq \mathbf{0}$, $\mathbf{x_2} \geq \mathbf{0}$ and $0 \leq \lambda \leq 1$. Hence $\mathbf{x_3}$ satisfies all the constraints of the given LPP. Thus x_3 is a feasible solution belonging to S. Hence S is convex.

Note: If an LPP has two feasible solutions then it has an infinite number of feasible solutions, as any convex combination of the two feasible solutions is a feasible solution.

Theorem 1.4: *The collection of all feasible solutions of an LPP constitutes a convex set whose extreme points correspond to the basic feasible solutions.*

Proof: Let us consider the LP problem

$$\text{Maximize} \quad z = \mathbf{cx}$$
$$\text{subject to} \ \ \mathbf{Ax} = \mathbf{b} \ \ \text{and} \ \ \mathbf{x} \geq \mathbf{0}$$

Then the feasible region S of the LPP is given by $S = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b},\ \ \mathbf{x} \geq \mathbf{0}\}$. Since S is a convex polyhedron, it is non-empty, closed and bounded. The objective function $z = \mathbf{cx},\ \ \mathbf{x} \in S$ which is non-empty, closed and bounded. Therefore, z attains its maximum on S. This proves the existence of an optimal solution.

Now, since S is a convex polyhedron, it has a finite number of extreme points. Let these be $\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_k} \in S$. Therefore, any $\mathbf{x} \in S$ can be expressed as a convex combination of the extreme points. Hence we can write

$$\mathbf{x} = \sum_{j=1}^{k} \alpha_j \mathbf{x_j}; \quad \alpha_j \geq 0 \ \ \text{and} \ \ \sum_{j=1}^{k} \alpha_j = 1.$$

Let $z_0 = \max\{\mathbf{cx_j},\ \ j = 1, 2, \cdots, k\}$. Then for any $\mathbf{x} \in S$,

$$z = \mathbf{cx} = \mathbf{c}(\sum_{j=1}^{k} \alpha_j \mathbf{x_j}) = \sum_{j=1}^{k} \alpha_j (\mathbf{cx_j}) \leq \sum_{j=1}^{k} \alpha_j z_0 = z_0$$

Therefore, $z \leq z_0$ for any $\mathbf{x} \in S$. Thus, the maximum value of z is attained only at one of the extreme points of S. That is, at least one extreme point of S yields an optimal solution. Since each extreme point of S corresponds to a basic feasible solution of the LPP, therefore, at least one basic feasible solution is optimal. This completes the proof.

1.3 Graphical Method

To solve an LPP, the graphical method is used when there are only two decision variables. If the problem has three or more variables then we use the simplex method which will be discussed in the next section.

Example 1.3: Solve the following LPP by graphical method:

$$\text{Minimize } Z = 20x_1 + 10x_2$$

subject to

$$x_1 + 2x_2 \leq 40$$
$$3x_1 + x_2 \geq 30$$
$$4x_1 + 3x_2 \geq 60$$
$$x_1, x_2 \geq 0.$$

Solution: Plot the graphs of all constraints by treating as linear equation. Then use the inequality constraints to mark the feasible region as shown by the shaded area in Fig. 1.1. This region is bounded below by the extreme points A(15,0), B(40,0), C(4,18) and D(6,12). The minimum value of the objective function occurs at the point D(6,12). Hence, the optimal solution to the given LPP is $x_1 = 6$, $x_2 = 12$ and $Z_{min} = 240$.

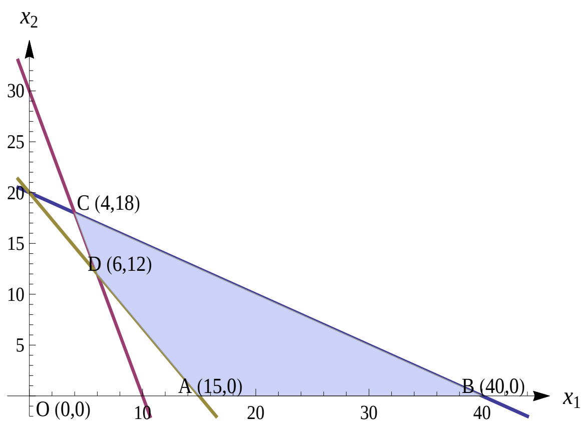

Fig. 1.1: Unique optimal solution in Example 1.3

Extreme point	Objective function $Z = 20x_1 + 10x_2$
A (15,0)	300
B (40,0)	800
C (4,18)	260
D (6,12)	240

Example 1.4: Solve the following LPP by graphical method:

$$\text{Maximize } Z = 4x_1 + 3x_2$$

subject to

$$x_1 + 2x_2 \leq 6$$
$$2x_1 + x_2 \leq 8$$
$$x_1 \geq 7$$
$$x_1, x_2 \geq 0.$$

Solution: The constraints are plotted on the graph as shown in Fig. 1.2. As there is no feasible region of solution space, the problem has no feasible solution.

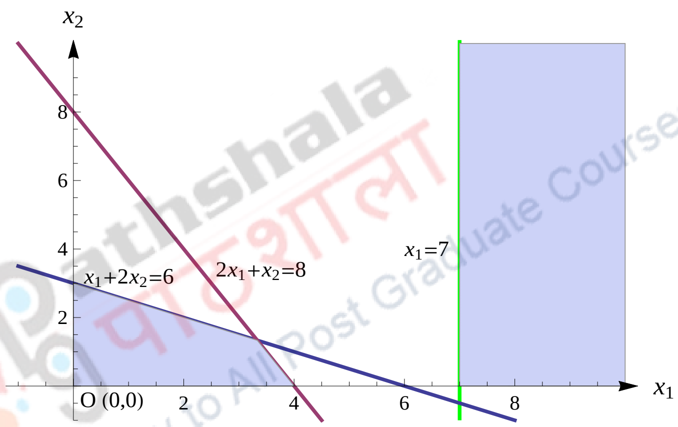

Fig. 1.2: No feasible solution in Example 1.4

Example 1.5: Show by graphical method that the following LPP has unbounded solution.

$$\text{Maximize } Z = 3x_1 + 5x_2$$

subject to

$$x_1 + 2x_2 \geq 10$$
$$x_1 \geq 5$$
$$x_2 \leq 10$$
$$x_1, x_2 \geq 0.$$

Solution: From the graph as shown in Fig. 1.3, it is clear that the feasible region is open-ended. Therefore, the value of Z can be made infinitely large without violating any of the constraints. Hence there exists an unbounded solution of the LPP.

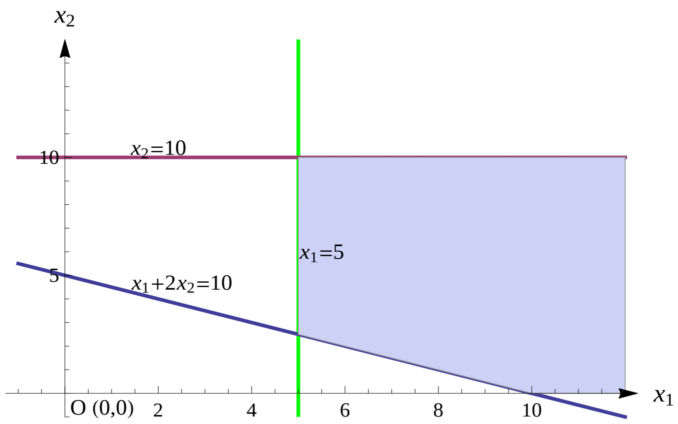

Fig. 1.3: Unbounded solution in Example 1.5

Note: Unbounded feasible region does not necessarily imply that no finite optimal solution of LP problem exists. Consider the following LPP which has an optimal feasible solution in spite of unbounded feasible region:

Maximize $Z = 2x_1 - x_2$

subject to

$$x_1 - x_2 \leq 1$$
$$x_1 \leq 3$$
$$x_1, x_2 \geq 0$$

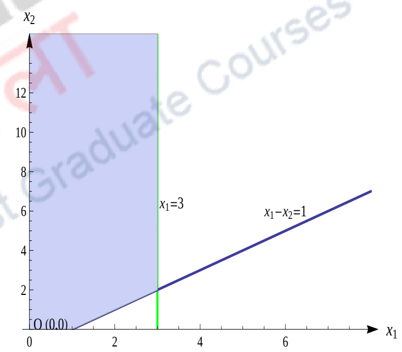

Fig. 1.4: Finite optimal solution

Example 1.6: Solve the following LP problem by graphical method:

Maxiimize $Z = 3x_1 + 2x_2$

subject to

$$6x_1 + 4x_2 \leq 24$$
$$x_2 \geq 2, \quad x_1 \leq 3, \quad x_1, x_2 \geq 0.$$

Solution: The constraints are plotted on a graph by treating as equations and then their inequality signs are used to identify feasible region as shown in Fig. 1.5.

The extreme points of the region are A(0,2), B(0,6), C(2,3) and D(2,2). The slope of the objective function and the first constraint equation $6x_1 + 4x_2 = 24$ coincide at line BC. Also, BC is the boundary line of the feasible region. This implies that an

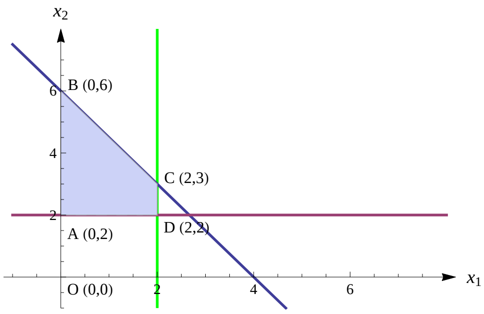

Fig. 1.5: An infinite number of optimal solutions in Example 1.6

Table 1.1

Corners (x, y)	Objective Function $Z = 3x_1 + 2x_2$
A (0,2)	4
B (0,6)	12
C (2,3)	12
D (2,2)	10

optimal solution of LP problem can be obtained at any point lies on the line segment BC. It is observed from Table 1.1 that the optimal value ($Z = 12$) is the same at two different extreme points B and C. Therefore, several combinations of any two points on the line segment BC give the same value of the objective function, which are also optimal solutions of the LP problem. Hence, there exists an infinite number of optimal solutions of the given LP problem.